

ON STRONG CONVERGENCE OF HALPERN'S METHOD USING AVERAGED TYPE MAPPINGS

F. CIANCIARUSO, G. MARINO, A. RUGIANO, B. SCARDAMAGLIA

ABSTRACT. In this paper, inspired by Iemoto and Takahashi [S. Iemoto, W. Takahashi, *Nonlinear Analysis* 71, (2009), 2082-2089], we study the Halpern's method to approximate strongly fixed points of a nonexpansive mapping and of a nonspreading mapping. A crucial tool in our results is the regularization with the averaged type mappings [C. Byrne, *Inverse Probl.* 20, (2004), 103-120].

1. INTRODUCTION

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, which induces the norm $\| \cdot \|$.

Let C be a nonempty, closed and convex subset of H . Let T be a nonlinear mapping of C into itself; we denote with $Fix(T)$ the set of fixed points of T , that is, $Fix(T) = \{z \in C : Tz = z\}$.

We recall that a mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

The problem of finding fixed points of nonexpansive mappings has been widely investigated by many authors.

For a fixed $u \in C$ and for each $t \in (0, 1)$, let z_t be the unique fixed point of the contraction given by

$$T_t x = tu + (1 - t)Tx, \quad x \in C.$$

Namely, we have $z_t = tu + (1 - t)Tz_t$. Browder [2] proved the following strong convergence theorem.

Theorem 1.1. *Let C be a nonempty, bounded, closed and convex subset of a Hilbert space H and let $T : C \rightarrow C$ be a nonexpansive mapping. Fix $u \in C$ and define $z_t \in C$ as $z_t = tu + (1 - t)Tz_t$ for $t \in (0, 1)$. Then as t tends to 0, z_t converges strongly to the unique element of $Fix(T)$ nearest to u , i.e. z_t converges strongly to $P_{Fix(T)}u$.*

We recall that if C is a nonempty closed convex subset of H , then for every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Such P_C is called the metric projection of H onto C .

If T is a nonexpansive mapping and $u \in C$ fixed, Halpern [5] was the first who considered the following explicit method:

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$$x_1 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1 \quad (1.1)$$

where $(\alpha_n)_{n \in \mathbb{N}} \subset [0, 1]$.

Moreover, Halpern proved in [5] the following Theorem on the convergence of (1.1) for a particular choice of $(\alpha_n)_{n \in \mathbb{N}}$.

Theorem 1.2. *Let C be a bounded, closed and convex subset of a Hilbert space H and let $T : C \rightarrow C$ be a nonexpansive mapping. For any initialization $x_1 \in C$ and anchor $u \in C$, define a sequence $(x_n)_{n \in \mathbb{N}}$ in C by*

$$x_{n+1} = n^{-\theta}u + (1 - n^{-\theta})Tx_n, \quad \forall n \geq 1,$$

where $\theta \in (0, 1)$. Then $(x_n)_{n \in \mathbb{N}}$ converges strongly to the element of $\text{Fix}(T)$ nearest to u .

He also showed that the control conditions

$$(C1) \quad \lim_{n \rightarrow \infty} \alpha_n = 0;$$

$$(C2) \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

are necessary for the convergence of (1.1) to a fixed point of T .

Subsequently, several authors carefully studied the following problem: are the control conditions (C1) and (C2) sufficient for the convergence of (1.1)?

In this direction, C.E. Chidume and C.O. Chidume [3] and Suzuki [14], independently, proved that the conditions (C1) and (C2) are sufficient to assure the strong convergence to a fixed point of T of the following iterative sequence:

$$x_1, u \in C; \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)(\lambda x_n + (1 - \lambda)Tx_n), \quad \forall n \geq 1.$$

Recently, in the setting of Banach spaces, Song and Chai [13], under the same conditions (C1) and (C2), but under stronger hypotheses on the mapping, obtained strong convergence of Halpern iterations (1.1). In particular, they assumed that E is a real reflexive Banach space with a uniformly Gateaux differentiable norm and with the fixed point property for nonexpansive self-mappings and considered an important subclass of nonexpansive mappings: the firmly type nonexpansive mappings.

Let T be a mapping with domain $D(T)$. T is said to be firmly type nonexpansive [13] if for all $x, y \in D(T)$, there exists $k \in (0, +\infty)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - k\|(x - Tx) - (y - Ty)\|^2.$$

A more general class of firmly type nonexpansive mappings is the class of the strongly nonexpansive mappings. Recall that a mapping $T : C \rightarrow C$ is said to be strongly nonexpansive if:

- (1) T is nonexpansive;
- (2) $x_n - y_n - (Tx_n - Ty_n) \rightarrow 0$, whenever $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are sequences in C such that $(x_n - y_n)_{n \in \mathbb{N}}$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \rightarrow 0$.

Saejung [12] proved the strong convergence of the Halpern's iterations (1.1) for strongly nonexpansive mappings in a Banach space E such that one of the following conditions is satisfied:

- E is uniformly smooth;
- E is reflexive, strictly convex with a uniformly Gateaux differentiable norm.

In the setting of Hilbert spaces, Kohsaka and Takahashi [8] defined $T : C \rightarrow C$ a nonspreading mapping if:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \quad \forall x, y \in C.$$

The following Lemma is an useful characterization of a nonspreading mapping.

Lemma 1.3. [7] *Let C be a nonempty closed subset of a Hilbert space H . Then a mapping $T : C \rightarrow C$ is nonspreading if and only if*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C. \quad (1.2)$$

Observe that if T is a nonspreading mapping from C into itself and $\text{Fix}(T) \neq \emptyset$, then T is quasi-nonexpansive, i.e.

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C, \quad \forall p \in \text{Fix}(T).$$

Further, the set of fixed points of a quasi-nonexpansive mapping is closed and convex [6].

Osilike and Isiogugu [11] studied the Halpern's type for k -strictly pseudononspreading mappings T , which are a more general class of the nonspreading mappings.

To obtain the strong convergence of (1.1) they replaced the mapping T with the averaged type mapping A_T , i.e. with the mapping:

$$A_T = (1 - \delta)I + \delta T, \quad \delta \in (0, 1).$$

Iemoto and Takahashi [7] approximated common fixed points of a nonexpansive mapping T and of a nonspreading mapping S in a Hilbert space using Moudafi's iterative scheme [10]. They obtained the following Theorem that states the weak convergence of their iterative method:

Theorem 1.4. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H . Assume that $\text{Fix}(S) \cap \text{Fix}(T) \neq \emptyset$. Define a sequence $(x_n)_{n \in \mathbb{N}}$ as follows:*

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[\beta_n Sx_n + (1 - \beta_n)Tx_n], \end{cases}$$

for all $n \in \mathbb{N}$, where $(\alpha_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}} \subset [0, 1]$. Then, the following hold:

- (i) If $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, then $(x_n)_{n \in \mathbb{N}}$ converges weakly to $p \in \text{Fix}(S)$;
- (ii) If $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $(x_n)_{n \in \mathbb{N}}$ converges weakly to $p \in \text{Fix}(T)$;
- (iii) If $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$, then $(x_n)_{n \in \mathbb{N}}$ converges weakly to $p \in \text{Fix}(S) \cap \text{Fix}(T)$.

In this paper, inspired by Iemoto and Takahashi [7], we introduce an iterative method of Halpern's type to approximate strongly fixed points of a nonexpansive mapping T and a nonspreading mapping S . A crucial tool to prove the strong convergence of our iterative scheme is the use of averaged type mappings A_T and A_S which have a regularizing role.

2. PRELIMINARIES

To begin, we collect some Lemmas which we use in our proofs in the next section. Let H be a real Hilbert space.

Lemma 2.1. *The following known results hold:*

- (1) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$,
for all $x, y \in H$ and for all $t \in [0, 1]$.
- (2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$,
for all $x, y \in H$.

The following Lemma [15] characterizes the projection P_C .

Lemma 2.2. *Let C be a closed and convex subset of a real Hilbert space and let P_C be the metric projection from H onto C . Given $x \in H$ and $z \in C$; then $z = P_C x$ if and only if there holds the inequality:*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C.$$

To prove our main Theorem, we need some fundamental properties of involved mappings.

The following result summarizes some significant properties of $I - T$ if T is a nonexpansive mapping ([1],[4]).

Lemma 2.3. *Let C be a nonempty closed convex subset of H and let $T : C \rightarrow C$ be nonexpansive. Then:*

- (1) $I - T : C \rightarrow H$ is $\frac{1}{2}$ -inverse strongly monotone, i.e.,

$$\frac{1}{2}\|(I - T)x - (I - T)y\|^2 \leq \langle x - y, (I - T)x - (I - T)y \rangle,$$

for all $x, y \in C$;

- (2) moreover, if $\text{Fix}(T) \neq \emptyset$, $I - T$ is demiclosed at 0, i.e. for every sequence $(x_n)_{n \in \mathbb{N}}$ weakly convergent to p such that $x_n - Tx_n \rightarrow 0$ as $n \rightarrow \infty$, it follows $p \in \text{Fix}(T)$.

If C is a nonempty, closed and convex subset of H and T is a nonlinear mapping of C into itself, inspired by [1], we can define the averaged type mapping as follows

$$A_T = (1 - \delta)I + \delta T = I - \delta(I - T) \quad (2.1)$$

where $\delta \in (0, 1)$. We notice that $\text{Fix}(T) = \text{Fix}(A_T)$ and that if T is a nonexpansive mapping also A_T is nonexpansive.

If S is a nonspreading mapping of C into itself and $\text{Fix}(S) \neq \emptyset$, we observe that A_S is quasi-nonexpansive and further the set of fixed points of A_S is closed and convex. The following Lemma shows the demiclosedness of $I - S$ at 0.

Lemma 2.4. [7] *Let C be a nonempty, closed and convex subset of H . Let $S : C \rightarrow C$ be a nonspreading mapping such that $\text{Fix}(S) \neq \emptyset$. Then $I - S$ is demiclosed at 0.*

In the sequel we use the following property of $I - S$.

Lemma 2.5. [7] *Let C be a nonempty, closed and convex subset of H . Let $S : C \rightarrow C$ be a nonspreading mapping. Then*

$$\|(I - S)x - (I - S)y\|^2 \leq \langle x - y, (I - S)x - (I - S)y \rangle + \frac{1}{2} \left(\|x - Sx\|^2 + \|y - Sy\|^2 \right),$$

for all $x, y \in C$.

If $\text{Fix}(S)$ is nonempty, Osilike and Isiogugu [11] proved that the averaged type mapping A_S is quasi-firmly type nonexpansive mapping, i.e. is a firmly type nonexpansive mapping on fixed points of S . On the same line of the proof in [11], we prove the following:

Proposition 2.6. *Let C be a nonempty closed and convex subset of H and let $S : C \rightarrow C$ be a nonspreading mapping such that $\text{Fix}(S)$ is nonempty. Then the averaged type mapping A_S*

$$A_S = (1 - \delta)I + \delta S, \quad (2.2)$$

is quasi-firmly type nonexpansive mapping with coefficient $k = (1 - \delta) \in (0, 1)$.

Proof. We obtain

$$\begin{aligned} \|A_S x - A_S y\|^2 &= \|(1 - \delta)(x - y) + \delta(Sx - Sy)\|^2 \\ (\text{by Lemma 2.1}) &= (1 - \delta)\|x - y\|^2 + \delta\|Sx - Sy\|^2 \\ &\quad - \delta(1 - \delta)\|(x - Sx) - (y - Sy)\|^2 \\ (\text{by (1.2)}) &\leq (1 - \delta)\|x - y\|^2 + \delta\left[\|x - y\|^2 + 2\langle x - Sx, y - Sy \rangle\right] \\ &\quad - \delta(1 - \delta)\|(x - Sx) - (y - Sy)\|^2 \\ &= \|x - y\|^2 + \frac{2}{\delta}\langle \delta(x - Sx), \delta(y - Sy) \rangle \\ &\quad - \frac{1 - \delta}{\delta}\|\delta(x - Sx) - \delta(y - Sy)\|^2 \\ (\text{by (2.2)}) &= \|x - y\|^2 + \frac{2}{\delta}\langle x - A_S x, y - A_S y \rangle \\ &\quad - \frac{1 - \delta}{\delta}\|(x - A_S x) - (y - A_S y)\|^2 \\ &\leq \|x - y\|^2 + \frac{2}{\delta}\langle x - A_S x, y - A_S y \rangle \\ &\quad - (1 - \delta)\|(x - A_S x) - (y - A_S y)\|^2. \end{aligned}$$

Hence, we have

$$\|A_S x - A_S y\|^2 \leq \|x - y\|^2 + \frac{2}{\delta}\langle x - A_S x, y - A_S y \rangle - (1 - \delta)\|(x - A_S x) - (y - A_S y)\|^2. \quad (2.3)$$

In particular, choosing in (2.3) $y = p$, where $p \in \text{Fix}(S) = \text{Fix}(A_S)$ we obtain

$$\|A_S x - p\|^2 \leq \|x - p\|^2 - (1 - \delta)\|x - A_S x\|^2. \quad (2.4)$$

□

A pertinent tool for us is the well-known Lemma of Xu [16].

Lemma 2.7. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 0,$$

where,

$$\bullet (\alpha_n)_{n \in \mathbb{N}} \subset [0, 1], \quad \sum_{n=1}^{\infty} \alpha_n = \infty;$$

- $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$;
- $\gamma_n \geq 0$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then,

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Finally, a crucial tool for our results is the following Lemma proved by Maingé.

Lemma 2.8. [9] *Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $(\gamma_{n_j})_{j \in \mathbb{N}}$ of $(\gamma_n)_{n \in \mathbb{N}}$ such that $\gamma_{n_j} < \gamma_{n_j+1}$, for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $(m_k)_{k \in \mathbb{N}}$ of \mathbb{N} such that $\lim_{k \rightarrow \infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$\gamma_{m_k} \leq \gamma_{m_k+1} \quad \text{and} \quad \gamma_k \leq \gamma_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, \dots, k\}$ such that the condition $\gamma_n < \gamma_{n+1}$ holds.

3. THE MAIN RESULT

Theorem 3.1. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H . Let $T : C \rightarrow C$ be a nonexpansive mapping and let $S : C \rightarrow C$ be a nonspreading mapping such that $\text{Fix}(S) \cap \text{Fix}(T) \neq \emptyset$. Let A_T and A_S be the averaged type mappings, i.e.*

$$A_T = (1 - \delta)I + \delta T, \quad A_S = (1 - \delta)I + \delta S, \quad \delta \in (0, 1).$$

Suppose that $(\alpha_n)_{n \in \mathbb{N}}$ is a real sequence in $(0, 1)$ satisfying the conditions:

- (1) $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If $(\beta_n)_{n \in \mathbb{N}}$ is a sequence in $[0, 1]$, we define a sequence $(x_n)_{n \in \mathbb{N}}$ as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)[\beta_n A_T x_n + (1 - \beta_n) A_S x_n], \quad n \in \mathbb{N}. \end{cases}$$

Then, the following hold:

- (i) *If $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, then $(x_n)_{n \in \mathbb{N}}$ converges strongly to $p \in \text{Fix}(T)$;*
- (ii) *If $\sum_{n=1}^{\infty} \beta_n < \infty$, then $(x_n)_{n \in \mathbb{N}}$ converges strongly to $p \in \text{Fix}(S)$;*
- (iii) *If $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$, then $(x_n)_{n \in \mathbb{N}}$ converges strongly to $p \in \text{Fix}(T) \cap \text{Fix}(S)$.*

Proof. We begin to prove that $(x_n)_{n \in \mathbb{N}}$ is bounded.

Put

$$U_n = \beta_n A_T + (1 - \beta_n) A_S. \tag{3.1}$$

Notice that U_n is quasi-nonexpansive, for all $n \in \mathbb{N}$.

For $q \in \text{Fix}(T) \cap \text{Fix}(S)$, we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n(u - q) + (1 - \alpha_n)(U_n x_n - q)\| \\ &\leq \alpha_n \|u - q\| + (1 - \alpha_n) \|U_n x_n - q\| \\ &\leq \alpha_n \|u - q\| + (1 - \alpha_n) \|x_n - q\| \end{aligned} \quad (3.2)$$

Since

$$\|x_1 - q\| \leq \max\{\|u - q\|, \|x_1 - q\|\},$$

and by induction we assume that

$$\|x_n - q\| \leq \max\{\|u - q\|, \|x_1 - q\|\},$$

then

$$\begin{aligned} \|x_{n+1} - q\| &\leq \alpha_n \|u - q\| + (1 - \alpha_n) \max\{\|u - q\|, \|x_1 - q\|\} \\ &\leq \alpha_n \max\{\|u - q\|, \|x_1 - q\|\} + (1 - \alpha_n) \max\{\|u - q\|, \|x_1 - q\|\} \\ &= \max\{\|u - q\|, \|x_1 - q\|\}. \end{aligned}$$

Thus $(x_n)_{n \in \mathbb{N}}$ is bounded. Consequently, $(A_T x_n)_{n \in \mathbb{N}}$, $(A_S x_n)_{n \in \mathbb{N}}$ and $(U_n x_n)_{n \in \mathbb{N}}$ are bounded as well.

Proof of (i) We introduce an auxiliary sequence

$$z_{n+1} = \alpha_n u + (1 - \alpha_n) A_T x_n, \quad n \in \mathbb{N}$$

and we study its properties and the relationship with the sequence $(x_n)_{n \in \mathbb{N}}$.

We shall divide the proof into several steps.

Step 1. $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.

Proof of Step 1. Observe that

$$\lim_{n \rightarrow \infty} \|z_{n+1} - A_T x_n\| = \lim_{n \rightarrow \infty} \alpha_n \|u - A_T x_n\| = 0. \quad (3.3)$$

Then we get

$$\begin{aligned} \|z_{n+1} - x_{n+1}\| &= \|\alpha_n u + (1 - \alpha_n) A_T x_n - \alpha_n u - (1 - \alpha_n) U_n x_n\| \\ &= (1 - \alpha_n) \|A_T x_n - U_n x_n\| \\ &= (1 - \alpha_n) \|A_T x_n - \beta_n A_T x_n - (1 - \beta_n) A_S x_n\| \\ &= (1 - \alpha_n)(1 - \beta_n) \|A_T x_n - A_S x_n\|. \end{aligned} \quad (3.4)$$

Since $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, we have

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \quad (3.5)$$

So, also $(z_n)_{n \in \mathbb{N}}$ is bounded.

Step 2. $\lim_{n \rightarrow \infty} \|z_n - A_T z_n\| = 0$.

Proof of Step 2. We begin to prove that $\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$.

Let $p \in \text{Fix}(T) = \text{Fix}(A_T)$. We have

$$\begin{aligned}
\|z_{n+1} - p\|^2 &= \|\alpha_n u + (1 - \alpha_n)(1 - \delta)x_n + (1 - \alpha_n)\delta T x_n - p\|^2 \\
&= \|(1 - \alpha_n)\delta(T x_n - x_n) + x_n - p\| + \alpha_n(u - x_n)\|^2 \\
(\text{ by Lemma 2.1}) &\leq \|(1 - \alpha_n)\delta(T x_n - x_n) + x_n - p\|^2 \\
&\quad + 2\alpha_n \langle u - x_n, z_{n+1} - p \rangle \\
&\leq (1 - \alpha_n)^2 \delta^2 \|T x_n - x_n\|^2 + \|x_n - p\|^2 \\
&\quad - 2(1 - \alpha_n)\delta \langle x_n - p, x_n - T x_n \rangle \\
&\quad + 2\alpha_n \|u - x_n\| \|z_{n+1} - p\| \\
&= (1 - \alpha_n)^2 \delta^2 \|x_n - T x_n\|^2 + \|x_n - p\|^2 \\
((I - T)p = 0) &- 2(1 - \alpha_n)\delta \langle x_n - p, (I - T)x_n - (I - T)p \rangle \\
&\quad + 2\alpha_n \|u - x_n\| \|z_{n+1} - p\| \\
(\text{ by Lemma 2.3}) &\leq \|x_n - p\|^2 + (1 - \alpha_n)^2 \delta^2 \|x_n - T x_n\|^2 \\
&\quad - (1 - \alpha_n)\delta \|(I - T)x_n - (I - T)p\|^2 \\
&\quad + 2\alpha_n \|u - x_n\| \|z_{n+1} - p\| \\
&= \|x_n - p\|^2 - (1 - \alpha_n)\delta[1 - \delta(1 - \alpha_n)] \|x_n - T x_n\|^2 \\
&\quad + 2\alpha_n \|u - x_n\| \|z_{n+1} - p\|
\end{aligned}$$

and hence

$$\begin{aligned}
&(1 - \alpha_n)\delta[1 - \delta(1 - \alpha_n)] \|x_n - T x_n\|^2 - 2\alpha_n \|u - x_n\| \|z_{n+1} - p\| \\
&\leq \|x_n - p\|^2 - \|z_{n+1} - p\|^2.
\end{aligned}$$

Set

$$L_n = (1 - \alpha_n)\delta[1 - \delta(1 - \alpha_n)] \|x_n - T x_n\|^2 - 2\alpha_n \|u - x_n\| \|z_{n+1} - p\|;$$

let us consider the following two cases.

a) If $L_n \leq 0$, for all $n \geq n_0$ large enough, then

$$(1 - \alpha_n)\delta[1 - \delta(1 - \alpha_n)] \|x_n - T x_n\|^2 \leq 2\alpha_n \|u - x_n\| \|z_{n+1} - p\|.$$

So, since $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} (1 - \alpha_n)\delta[1 - \delta(1 - \alpha_n)] = \delta(1 - \delta)$,

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0.$$

b) Assume now that there exists a subsequence $(L_{n_k})_{k \in \mathbb{N}}$ of $(L_n)_{n \in \mathbb{N}}$ taking all its positive terms; so $L_{n_k} > 0$ for every $k \in \mathbb{N}$, then

$$0 < L_{n_k} \leq \|x_{n_k} - p\|^2 - \|z_{n_k+1} - p\|^2. \quad (3.6)$$

Summing (3.6) from $k = 1$ to N , we obtain

$$\begin{aligned}
\sum_{k=1}^N L_{n_k} &\leq \|x_{n_1} - p\|^2 + \sum_{k=1}^{N-1} (\|x_{n_{k+1}} - p\|^2 - \|z_{n_{k+1}} - p\|^2) - \|z_{n_N+1} - p\|^2 \\
&\leq \|x_{n_1} - p\|^2 + \sum_{k=1}^{N-1} (\|x_{n_{k+1}} - p\| + \|z_{n_{k+1}} - p\|) \|x_{n_{k+1}} - z_{n_{k+1}}\| \\
&\leq \|x_{n_1} - p\|^2 \\
(\text{by (3.4)}) \quad &+ \sum_{k=1}^{N-1} (1 - \alpha_{n_k})(1 - \beta_{n_k})(\|x_{n_{k+1}} - p\| + \|z_{n_{k+1}} - p\|) \|A_T x_{n_k} - A_S x_{n_k}\| \\
&\leq \|x_{n_1} - p\|^2 + K \sum_{k=1}^{N-1} (1 - \beta_{n_k}),
\end{aligned}$$

where $K = \sup_{k \in \mathbb{N}} \{(\|x_{n_{k+1}} - p\| + \|z_{n_{k+1}} - p\|) \|A_T x_{n_k} - A_S x_{n_k}\|\}$.

Since $\sum_{k=1}^{\infty} (1 - \beta_{n_k}) < \infty$,

$$\sum_{k=1}^{\infty} ((1 - \alpha_{n_k})\delta[1 - \delta(1 - \alpha_{n_k})] \|x_{n_k} - T x_{n_k}\|^2 - 2\alpha_{n_k} \|u - x_{n_k}\| \|z_{n_{k+1}} - p\|) < \infty.$$

Thus,

$$\lim_{k \rightarrow \infty} ((1 - \alpha_{n_k})\delta[1 - \delta(1 - \alpha_{n_k})] \|x_{n_k} - T x_{n_k}\|^2 - 2\alpha_{n_k} \|u - x_{n_k}\| \|z_{n_{k+1}} - p\|) = 0,$$

and since $\lim_{k \rightarrow \infty} \alpha_{n_k} = 0$ and $\lim_{k \rightarrow \infty} (1 - \alpha_{n_k})\delta[1 - \delta(1 - \alpha_{n_k})] = \delta(1 - \delta)$, also in this case we get

$$\lim_{k \rightarrow \infty} \|x_{n_k} - T x_{n_k}\| = 0.$$

Since the remanent terms of the sequence $\|x_n - T x_n\|$ are not positive, from the case a) we can conclude that

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0.$$

Consequently,

$$\begin{aligned}
\lim_{n \rightarrow \infty} \|x_n - A_T x_n\| &= \lim_{n \rightarrow \infty} \|x_n - (1 - \delta)x_n - \delta T x_n\| \\
&= \lim_{n \rightarrow \infty} \delta \|x_n - T x_n\| = 0.
\end{aligned} \tag{3.7}$$

Furthermore, from (3.5) and (3.7) and

$$\begin{aligned}
\|z_n - A_T z_n\| &\leq \|z_n - x_n\| + \|x_n - A_T x_n\| + \|A_T x_n - A_T z_n\| \\
&\leq \|z_n - x_n\| + \|x_n - A_T x_n\| + \|x_n - z_n\|
\end{aligned}$$

we get

$$\lim_{n \rightarrow \infty} \|z_n - A_T z_n\| = 0. \tag{3.8}$$

Now, define the real sequence

$$t_n = \sqrt{\|z_n - A_T z_n\|}, \quad n \in \mathbb{N}. \tag{3.9}$$

Let $z_{t_n} \in C$ be the unique fixed point of the contraction V_{t_n} defined su C by

$$V_{t_n}x = t_n u + (1 - t_n)A_T x. \quad (3.10)$$

>From Browder's Theorem 1.1, $\lim_{n \rightarrow \infty} z_{t_n} = p_0 \in \text{Fix}(A_T)$; now we prove that:

Step 3. $\limsup_{n \rightarrow \infty} \langle u - p_0, z_n - p_0 \rangle \leq 0$.

Proof of Step 3. From (3.10), we have

$$z_{t_n} - z_n = t_n(u - z_n) + (1 - t_n)(A_T z_{t_n} - z_n).$$

We compute

$$\begin{aligned} \|z_{t_n} - z_n\|^2 &= \|t_n(u - z_n) + (1 - t_n)(A_T z_{t_n} - z_n)\|^2 \\ (\text{by Lemma 2.1}) &\leq (1 - t_n)^2 \|A_T z_{t_n} - z_n\|^2 + 2t_n \langle u - z_n, z_{t_n} - z_n \rangle \\ &\leq (1 - t_n)^2 (\|A_T z_{t_n} - A_T z_n\| + \|A_T z_n - z_n\|)^2 \\ &\quad + 2t_n \langle u - z_n, z_{t_n} - z_n \rangle \\ &= (1 - t_n)^2 [\|A_T z_{t_n} - A_T z_n\|^2 + \|A_T z_n - z_n\|^2 \\ &\quad + 2\|A_T z_n - z_n\| \|A_T z_{t_n} - A_T z_n\|] \\ &\quad + 2t_n \langle u - z_n, z_{t_n} - z_n \rangle + 2t_n \langle z_{t_n} - z_n, z_{t_n} - z_n \rangle \\ (A_T \text{ nonexpansive}) &\leq (1 - t_n)^2 [\|z_{t_n} - z_n\|^2 + \|A_T z_n - z_n\|^2 \\ &\quad + 2\|A_T z_n - z_n\| \|z_{t_n} - z_n\|] \\ &\quad + 2t_n \|z_{t_n} - z_n\|^2 + 2t_n \langle u - z_n, z_{t_n} - z_n \rangle \\ &= (1 + t_n^2) \|z_{t_n} - z_n\|^2 \\ &\quad + \|A_T z_n - z_n\| (\|A_T z_n - z_n\| + 2\|z_{t_n} - z_n\|) \\ &\quad + 2t_n \langle u - z_n, z_{t_n} - z_n \rangle. \end{aligned}$$

Hence

$$\begin{aligned} \langle u - z_{t_n}, z_n - z_{t_n} \rangle &\leq \frac{t_n}{2} \|z_{t_n} - z_n\|^2 \\ &\quad + \frac{\|A_T z_n - z_n\|}{2t_n} (\|A_T z_n - z_n\| + 2\|z_{t_n} - z_n\|). \end{aligned}$$

>From (3.9) and by the boundedness of $(z_{t_n})_{n \in \mathbb{N}}$, $(z_n)_{n \in \mathbb{N}}$ and $(A_T z_n)_{n \in \mathbb{N}}$ we have

$$\limsup_{n \rightarrow \infty} \langle u - z_{t_n}, z_n - z_{t_n} \rangle \leq 0. \quad (3.11)$$

Furthermore,

$$\begin{aligned} \langle u - z_{t_n}, z_n - z_{t_n} \rangle &= \langle u - p_0, z_n - z_{t_n} \rangle + \langle p_0 - z_{t_n}, z_n - z_{t_n} \rangle \\ &= \langle u - p_0, z_n - p_0 \rangle + \langle u - p_0, p_0 - z_{t_n} \rangle + \langle p_0 - z_{t_n}, z_n - z_{t_n} \rangle \end{aligned} \quad (3.12)$$

Since $\lim_{n \rightarrow \infty} z_{t_n} = p_0 \in \text{Fix}(A_T)$, we get

$$\lim_{n \rightarrow \infty} \langle p_0 - z_{t_n}, z_n - z_{t_n} \rangle = \lim_{n \rightarrow \infty} \langle u - p_0, p_0 - z_{t_n} \rangle = 0. \quad (3.13)$$

We conclude from (3.11), (3.12) and (3.13)

$$\limsup_{n \rightarrow \infty} \langle u - p_0, z_n - p_0 \rangle \leq 0. \quad (3.14)$$

Step 4. $(z_n)_{n \in \mathbb{N}}$ converges strongly to $p_0 \in \text{Fix}(T)$.

Proof of Step 4. We compute

$$\begin{aligned}
\|z_{n+1} - p_0\|^2 &= \|\alpha_n(u - p_0) + (1 - \alpha_n)(A_T x_n - p_0)\|^2 \\
(\text{by Lemma 2.1}) &\leq (1 - \alpha_n)^2 \|A_T x_n - p_0\|^2 + 2\alpha_n \langle u - p_0, z_{n+1} - p_0 \rangle \\
(A_T \text{ nonexpansive}) &\leq (1 - \alpha_n) \|x_n - p_0\|^2 + 2\alpha_n \langle u - p_0, z_{n+1} - p_0 \rangle \\
&\leq (1 - \alpha_n) (\|x_n - z_n\| + \|z_n - p_0\|)^2 \\
&\quad + 2\alpha_n \langle u - p_0, z_{n+1} - p_0 \rangle \\
&\leq (1 - \alpha_n) \|x_n - z_n\|^2 + (1 - \alpha_n) \|z_n - p_0\|^2 \\
&\quad + 2(1 - \alpha_n) \|x_n - z_n\| \|z_n - p_0\| \\
&\quad + 2\alpha_n \langle u - p_0, z_{n+1} - p_0 \rangle \\
&\leq (1 - \alpha_n) \|z_n - p_0\|^2 \\
(\text{by (3.4)}) &\quad + (1 - \alpha_n)(1 - \alpha_{n-1})^2(1 - \beta_{n-1})^2 \|A_T x_{n-1} - A_S x_{n-1}\|^2 \\
&\quad + 2(1 - \alpha_n)(1 - \alpha_{n-1})(1 - \beta_{n-1}) \|A_T x_{n-1} - A_S x_{n-1}\| \|z_n - p_0\| \\
&\quad + 2\alpha_n \langle u - p_0, z_{n+1} - p_0 \rangle \\
&\leq (1 - \alpha_n) \|z_n - p_0\|^2 + (1 - \beta_{n-1}) \|A_T x_{n-1} - A_S x_{n-1}\|^2 \\
&\quad + 2(1 - \beta_{n-1}) \|A_T x_{n-1} - A_S x_{n-1}\| \|z_n - p_0\| \\
&\quad + 2\alpha_n \langle u - p_0, z_{n+1} - p_0 \rangle \\
&\leq (1 - \alpha_n) \|z_n - p_0\|^2 + M(1 - \beta_{n-1}) \\
&\quad + 2\alpha_n \langle u - p_0, z_{n+1} - p_0 \rangle,
\end{aligned}$$

where $M := \sup_{n \in \mathbb{N}} \{ \|A_T x_{n-1} - A_S x_{n-1}\|^2 + 2 \|A_T x_{n-1} - A_S x_{n-1}\| \|z_n - p_0\| \}$.

Since by hypothesis $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, from (3.14) we can apply Lemma 2.7 and conclude that

$$\lim_{n \rightarrow \infty} \|z_{n+1} - p_0\| = 0.$$

By $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$, we have

$$\lim_{n \rightarrow \infty} \|x_n - p_0\| = 0.$$

Hence, $(x_n)_{n \in \mathbb{N}}$ converges strongly to $p_0 \in \text{Fix}(T)$. \square

Proof of (ii)

Again we introduce an other auxiliary sequence

$$s_{n+1} = \alpha_n u + (1 - \alpha_n) A_S x_n, \quad (3.15)$$

and we study its properties and the relationship with the sequence $(x_n)_{n \in \mathbb{N}}$.

Recall that $A_S = (1 - \delta)I + \delta S$, with $\delta \in (0, 1)$.

We shall divide the proof into several steps.

Proof. Step 1. $\lim_{n \rightarrow \infty} \|x_n - s_n\| = 0$.

Proof of Step 1. We observe that

$$\lim_{n \rightarrow \infty} \|s_{n+1} - A_S x_n\| = \lim_{n \rightarrow \infty} \alpha_n \|u - A_S x_n\| = 0. \quad (3.16)$$

We compute

$$\begin{aligned}
\|x_{n+1} - s_{n+1}\| &= \|\alpha_n u + (1 - \alpha_n)Ux_n - \alpha_n u - (1 - \alpha_n)A_S x_n\| \\
&= (1 - \alpha_n)\|Ux_n - A_S x_n\| \\
&= (1 - \alpha_n)\|\beta_n A_T x_n + (1 - \beta_n)A_S x_n - A_S x_n\| \\
&= (1 - \alpha_n)\beta_n\|A_T x_n - A_S x_n\|.
\end{aligned} \tag{3.17}$$

Since $\sum_{n=1}^{\infty} \beta_n < \infty$,

$$\lim_{n \rightarrow \infty} \|x_n - s_n\| = 0. \tag{3.18}$$

This shows that also $(s_n)_{n \in \mathbb{N}}$ is bounded.

Step 2. $\lim_{n \rightarrow \infty} \|x_n - A_S x_n\| = 0$.

Proof of Step 2. We begin to prove that $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$.

Let $p \in \text{Fix}(S) = \text{Fix}(A_S)$. We compute

$$\begin{aligned}
\|s_{n+1} - p\|^2 &= \|\alpha_n u + (1 - \alpha_n)(1 - \delta)x_n + (1 - \alpha_n)\delta Sx_n - p\|^2 \\
&= \|[(1 - \alpha_n)\delta(Sx_n - x_n) + x_n - p] + \alpha_n(u - x_n)\|^2 \\
(\text{ by Lemma 2.1}) &\leq \|(1 - \alpha_n)\delta(Sx_n - x_n) + x_n - p\|^2 \\
&\quad + 2\alpha_n \langle u - x_n, s_{n+1} - p \rangle \\
&\leq (1 - \alpha_n)^2 \delta^2 \|Sx_n - x_n\|^2 + \|x_n - p\|^2 \\
&\quad - 2(1 - \alpha_n)\delta \langle x_n - p, x_n - Sx_n \rangle \\
&\quad + 2\alpha_n \|u - x_n\| \|s_{n+1} - p\| \\
&= (1 - \alpha_n)^2 \delta^2 \|x_n - Sx_n\|^2 + \|x_n - p\|^2 \\
((I - S)p = 0) &- 2(1 - \alpha_n)\delta \langle x_n - p, (I - S)x_n - (I - S)p \rangle \\
&\quad + 2\alpha_n \|u - x_n\| \|s_{n+1} - p\| \\
(\text{ by Lemma 2.5}) &\leq \|x_n - p\|^2 + (1 - \alpha_n)^2 \delta^2 \|x_n - Sx_n\|^2 \\
&\quad - 2(1 - \alpha_n)\delta \left[\|(I - S)x_n - (I - S)p\|^2 \right. \\
&\quad \left. - \frac{1}{2} \left(\|x_n - Sx_n\|^2 + \|p - Sp\|^2 \right) \right] \\
&\quad + 2\alpha_n \|u - x_n\| \|s_{n+1} - p\| \\
&= \|x_n - p\|^2 + (1 - \alpha_n)^2 \delta^2 \|x_n - Sx_n\|^2 \\
&\quad - (1 - \alpha_n)\delta \|x_n - Sx_n\|^2 + 2\alpha_n \|u - x_n\| \|s_{n+1} - p\| \\
&= \|x_n - p\|^2 - (1 - \alpha_n)\delta [1 - \delta(1 - \alpha_n)] \|x_n - Sx_n\|^2 \\
&\quad + 2\alpha_n \|u - x_n\| \|s_{n+1} - p\|
\end{aligned}$$

and hence

$$\begin{aligned}
&(1 - \alpha_n)\delta [1 - \delta(1 - \alpha_n)] \|x_n - Sx_n\|^2 - 2\alpha_n \|u - x_n\| \|s_{n+1} - p\| \\
&\leq \|x_n - p\|^2 - \|s_{n+1} - p\|^2.
\end{aligned}$$

Set

$$L_n = (1 - \alpha_n)\delta [1 - \delta(1 - \alpha_n)] \|x_n - Sx_n\|^2 - 2\alpha_n \|u - x_n\| \|s_{n+1} - p\|;$$

let us consider the following two cases.

a) If $L_n \leq 0$, for all $n \geq n_0$ large enough, then

$$(1 - \alpha_n)\delta[1 - \delta(1 - \alpha_n)]\|x_n - Sx_n\|^2 \leq 2\alpha_n\|u - x_n\|\|s_{n+1} - p\|.$$

So, since $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} (1 - \alpha_n)\delta[1 - \delta(1 - \alpha_n)] = \delta(1 - \delta)$,

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0.$$

b) Assume now that there exists a subsequence $(L_{n_k})_{k \in \mathbb{N}}$ of $(L_n)_{n \in \mathbb{N}}$ taking all its positive terms; so $L_{n_k} > 0$ for every $k \in \mathbb{N}$, then

$$0 < L_{n_k} \leq \|x_{n_k} - p\|^2 - \|s_{n_k+1} - p\|^2. \quad (3.19)$$

Summing (3.19) from $k = 1$ to N , we obtain

$$\begin{aligned} \sum_{k=1}^N L_{n_k} &\leq \|x_{n_1} - p\|^2 + \sum_{k=1}^{N-1} (\|x_{n_{k+1}} - p\|^2 - \|s_{n_{k+1}} - p\|^2) - \|s_{n_N+1} - p\|^2 \\ &\leq \|x_{n_1} - p\|^2 + \sum_{k=1}^{N-1} (\|x_{n_{k+1}} - p\| + \|s_{n_{k+1}} - p\|)\|x_{n_{k+1}} - s_{n_{k+1}}\| \\ &\leq \|x_{n_1} - p\|^2 \\ (\text{by (3.17)}) \quad &+ \sum_{k=1}^{N-1} (1 - \alpha_{n_k})\beta_{n_k} (\|x_{n_{k+1}} - p\| + \|s_{n_{k+1}} - p\|)\|A_T x_{n_k} - A_S x_{n_k}\| \\ &\leq \|x_{n_1} - p\|^2 + K \sum_{k=1}^{N-1} \beta_{n_k}, \end{aligned}$$

where $K = \sup_{k \in \mathbb{N}} \{(\|x_{n_{k+1}} - p\| + \|s_{n_{k+1}} - p\|)\|A_T x_{n_k} - A_S x_{n_k}\|\}$.

Since $\sum_{k=1}^{\infty} \beta_{n_k} < \infty$,

$$\sum_{k=1}^{\infty} ((1 - \alpha_{n_k})\delta[1 - \delta(1 - \alpha_{n_k})]\|x_{n_k} - Sx_{n_k}\|^2 - 2\alpha_{n_k}\|u - x_{n_k}\|\|s_{n_{k+1}} - p\|) < \infty.$$

Thus,

$$\lim_{k \rightarrow \infty} ((1 - \alpha_{n_k})\delta[1 - \delta(1 - \alpha_{n_k})]\|x_{n_k} - Sx_{n_k}\|^2 - 2\alpha_{n_k}\|u - x_{n_k}\|\|s_{n_{k+1}} - p\|) = 0,$$

and since $\lim_{k \rightarrow \infty} \alpha_{n_k} = 0$ and $\lim_{k \rightarrow \infty} (1 - \alpha_{n_k})\delta[1 - \delta(1 - \alpha_{n_k})] = \delta(1 - \delta)$, also in this case we get

$$\lim_{k \rightarrow \infty} \|x_{n_k} - Sx_{n_k}\| = 0.$$

As in i), we can conclude that

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0.$$

Consequently,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - A_S x_n\| &= \lim_{n \rightarrow \infty} \|x_n - (1 - \delta)x_n - \delta Sx_n\| \\ &= \lim_{n \rightarrow \infty} \delta \|x_n - Sx_n\| = 0. \end{aligned} \quad (3.20)$$

Moreover, from (3.18) and (3.20),

$$\lim_{n \rightarrow \infty} \|s_n - A_S s_n\| = \lim_{n \rightarrow \infty} \delta \|s_n - Ss_n\| = 0.$$

Step 3. $\limsup_{n \rightarrow \infty} \langle u - P_{Fix(S)}u, s_n - P_{Fix(S)}u \rangle \leq 0$.

Proof of Step 4. We may assume without loss of generality that there exists a subsequence $(s_{n_j})_{j \in \mathbb{N}}$ of $(s_n)_{n \in \mathbb{N}}$ such that $s_{n_j} \rightarrow v$ and

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - P_{Fix(S)}u, s_n - P_{Fix(S)}u \rangle &= \lim_{j \rightarrow \infty} \langle u - P_{Fix(S)}u, s_{n_j} - P_{Fix(S)}u \rangle \\ &= \langle u - P_{Fix(S)}u, v - P_{Fix(S)}u \rangle \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|s_n - Ss_n\| = 0$ and from $I - S$ is demiclosed at 0, $v \in Fix(S) = Fix(A_S)$. Then by (2.2), we have

$$\limsup_{n \rightarrow \infty} \langle u - P_{Fix(S)}u, s_n - P_{Fix(S)}u \rangle = \langle u - P_{Fix(S)}u, v - P_{Fix(S)}u \rangle \leq 0. \quad (3.21)$$

Step 4. $(s_n)_{n \in \mathbb{N}}$ converges strongly to $P_{Fix(S)}u$.

Proof of Step 5. We compute

$$\begin{aligned} \|s_{n+1} - P_{Fix(S)}u\|^2 &= \|\alpha_n(u - P_{Fix(S)}u) + (1 - \alpha_n)(A_S x_n - P_{Fix(S)}u)\|^2 \\ (\text{by Lemma 2.1}) &\leq (1 - \alpha_n)^2 \|A_S x_n - P_{Fix(S)}u\|^2 \\ &\quad + 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle \\ (A_S \text{ nonexpansive}) &\leq (1 - \alpha_n) \|x_n - P_{Fix(S)}u\|^2 \\ &\quad + 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle \\ &\leq (1 - \alpha_n) (\|x_n - s_n\| + \|s_n - P_{Fix(S)}u\|)^2 \\ &\quad + 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle \\ &\leq (1 - \alpha_n) \|x_n - s_n\|^2 + (1 - \alpha_n) \|s_n - P_{Fix(S)}u\|^2 \\ &\quad + 2(1 - \alpha_n) \|x_n - s_n\| \|s_n - P_{Fix(S)}u\| \\ &\quad + 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle \\ &\leq (1 - \alpha_n) \|s_n - P_{Fix(S)}u\|^2 \\ \text{by (3.17)} &\quad + (1 - \alpha_n)(1 - \alpha_{n-1})^2 \beta_{n-1}^2 \|A_T x_{n-1} - A_S x_{n-1}\|^2 \\ &\quad + 2(1 - \alpha_n)(1 - \alpha_{n-1}) \beta_{n-1} \|A_T x_{n-1} - A_S x_{n-1}\| \|s_n - P_{Fix(S)}u\| \\ &\quad + 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle \\ &\leq (1 - \alpha_n) \|s_n - P_{Fix(S)}u\|^2 + \beta_{n-1} \|A_T x_{n-1} - A_S x_{n-1}\|^2 \\ &\quad + 2\beta_{n-1} \|A_T x_{n-1} - A_S x_{n-1}\| \|s_n - P_{Fix(S)}u\| \\ &\quad + 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle \\ &\leq (1 - \alpha_n) \|s_n - P_{Fix(S)}u\|^2 + M \beta_{n-1} \\ &\quad + 2\alpha_n \langle u - P_{Fix(S)}u, s_{n+1} - P_{Fix(S)}u \rangle, \end{aligned}$$

where $M := \sup_{n \in \mathbb{N}} \{\|A_T x_{n-1} - A_S x_{n-1}\|^2 + 2\|A_T x_{n-1} - A_S x_{n-1}\| \|s_n - P_{Fix(S)}u\|\}$.

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$ and from (3.21) we can apply Lemma 2.7 and we conclude that

$$\lim_{n \rightarrow \infty} \|s_{n+1} - P_{Fix(S)}u\| = 0.$$

So, $(s_n)_{n \in \mathbb{N}}$ converges strongly to $P_{Fix(S)}u \in Fix(S)$. Since $\lim_{n \rightarrow \infty} \|x_n - s_n\| = 0$, we have

$$\lim_{n \rightarrow \infty} \|x_n - P_{Fix(S)}u\| = 0,$$

i.e. $(x_n)_{n \in \mathbb{N}}$ converges strongly to $P_{\text{Fix}(S)}u \in \text{Fix}(S)$. \square

Proof of (iii)

Proof. Let $q \in \text{Fix}(S) \cap \text{Fix}(T)$.

Since in this last case, the techniques used in i) and ii) fail, we turn our attention on the monotony of the sequence $(\|x_n - q\|)_{n \in \mathbb{N}}$. We consider the following two cases.

Case 1. $\|x_{n+1} - q\| \leq \|x_n - q\|$, for every $n \geq n_0$ large enough.

Case 2. There exists a subsequence $(\|x_{n_j} - q\|)_{j \in \mathbb{N}}$ of $(\|x_n - q\|)_{n \in \mathbb{N}}$ such that

$$\|x_{n_j} - q\| < \|x_{n_j+1} - q\| \text{ for all } j \in \mathbb{N}.$$

Case 1. $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists finite and hence

$$\lim_{n \rightarrow \infty} (\|x_{n+1} - q\| - \|x_n - q\|) = 0. \quad (3.22)$$

We shall divide the proof into several steps.

Step 1. $\lim_{n \rightarrow \infty} \|x_n - A_S x_n\| = 0$.

Proof of Step 1. Consider

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \beta_n (A_T x_n + (1 - \beta_n) A_S x_n). \quad (3.23)$$

We compute

$$\begin{aligned} \|\beta_n (A_T x_n - q) + (1 - \beta_n) (A_S x_n - q)\|^2 &= \beta_n \|A_T x_n - q\|^2 \\ &\quad + (1 - \beta_n) \|A_S x_n - q\|^2 \\ &\quad - \beta_n (1 - \beta_n) \|A_T x_n - A_S x_n\|^2 \\ (A_T \text{ nonexpansive and by (2.4)}) &\leq \beta_n \|x_n - q\|^2 + (1 - \beta_n) \|x_n - q\|^2 \\ &\quad - (1 - \beta_n) (1 - \delta) \|x_n - A_S x_n\|^2 \\ &\quad - \beta_n (1 - \beta_n) \|A_T x_n - A_S x_n\|^2 \\ &= \|x_n - q\|^2 - (1 - \beta_n) (1 - \delta) \|x_n - A_S x_n\|^2 \\ &\quad - \beta_n (1 - \beta_n) \|A_T x_n - A_S x_n\|^2 \end{aligned}$$

We recall that $U_n = \beta_n A_T + (1 - \beta_n) A_S$.

So, we get

$$\|U_n x_n - q\|^2 \leq \|x_n - q\|^2 - (1 - \beta_n) (1 - \delta) \|x_n - A_S x_n\|^2 - \beta_n (1 - \beta_n) \|A_T x_n - A_S x_n\|^2. \quad (3.24)$$

We have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|U_n x_n - q + \alpha_n (u - U_n x_n)\|^2 \\ &\leq \|U_n x_n - q\|^2 + \alpha_n (\alpha_n \|u - U_n x_n\|^2 + 2 \|U_n x_n - q\| \|u - U_n x_n\|) \\ (\text{by (3.24)}) &\leq \|x_n - q\|^2 - (1 - \beta_n) (1 - \delta) \|x_n - A_S x_n\|^2 \\ &\quad - \beta_n (1 - \beta_n) \|A_T x_n - A_S x_n\|^2 + \alpha_n M, \end{aligned} \quad (3.25)$$

where $M := \sup_{n \in \mathbb{N}} \{\alpha_n \|u - U_n x_n\|^2 + 2 \|U_n x_n - q\| \|u - U_n x_n\|\}$. From (3.25),

we derive

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - (1 - \beta_n) (1 - \delta) \|x_n - A_S x_n\|^2 + \alpha_n M,$$

hence

$$(1 - \beta_n)(1 - \delta)\|x_n - A_S x_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \alpha_n M. \quad (3.26)$$

> From (3.22) and $\lim_{n \rightarrow \infty} \alpha_n = 0$, we get

$$\lim_{n \rightarrow \infty} ((1 - \beta_n)(1 - \delta)\|x_n - A_S x_n\|^2) = 0.$$

Since $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$, we have

$$\lim_{n \rightarrow \infty} \|x_n - A_S x_n\| = \lim_{n \rightarrow \infty} \delta \|x_n - S x_n\| = 0. \quad (3.27)$$

Step 2. $\lim_{n \rightarrow \infty} \|A_T x_n - A_S x_n\| = 0$.

Proof of Step 2. Moreover, from (3.25), we also can derive

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \beta_n(1 - \beta_n)\|A_T x_n - A_S x_n\|^2 + \alpha_n M,$$

hence

$$\beta_n(1 - \beta_n)\|A_T x_n - A_S x_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \alpha_n M.$$

As above, we can conclude that

$$\lim_{n \rightarrow \infty} \|A_T x_n - A_S x_n\| = \lim_{n \rightarrow \infty} \delta \|T x_n - S x_n\| = 0. \quad (3.28)$$

> From (3.27) and (3.28), it follows that

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0. \quad (3.29)$$

Let $F = \text{Fix}(T) \cap \text{Fix}(S)$.

Step 3. $\limsup_{n \rightarrow \infty} \langle u - P_F u, x_n - P_F u \rangle \leq 0$.

Proof of Step 3. We may assume without loss of generality that there exists a subsequence $(x_{n_j})_{j \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that $x_{n_j} \rightharpoonup v$ and

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - P_F u, x_n - P_F u \rangle &= \lim_{j \rightarrow \infty} \langle u - P_F u, x_{n_j} - P_F u \rangle \\ &= \langle u - P_F u, v - P_F u \rangle. \end{aligned} \quad (3.30)$$

By (3.29) and (3.27) and by the demiclosedness of $I - T$ at 0 and of $I - S$ at 0, $v \in F = \text{Fix}(T) \cap \text{Fix}(S)$. Then we can conclude that

$$\limsup_{n \rightarrow \infty} \langle u - P_F u, x_n - P_F u \rangle = \langle u - P_F u, v - P_F u \rangle \leq 0.$$

Step 4. $(x_n)_{n \in \mathbb{N}}$ converges strongly to $P_F u$.

Proof of Step 4. We compute

$$\begin{aligned} \|x_{n+1} - P_F u\|^2 &= \|\alpha_n(u - P_F u) + (1 - \alpha_n)(U_n x_n - P_F u)\|^2 \\ (\text{by Lemma 2.1}) &\leq (1 - \alpha_n)^2 \|U_n x_n - P_F u\|^2 \\ &\quad + 2\alpha_n \langle u - P_F u, x_{n+1} - P_F u \rangle \\ (U_n \text{ quasi-nonexpansive}) &\leq (1 - \alpha_n) \|x_n - P_F u\|^2 \\ &\quad + 2\alpha_n \langle u - P_F u, x_{n+1} - P_F u \rangle. \end{aligned} \quad (3.31)$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, we can apply Lemma 2.7 and conclude that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - P_F u\| = 0.$$

Finally, $(x_n)_{n \in \mathbb{N}}$ converges strongly to $P_F u$.

Case 2. Let $q = P_F u$. Then there exists a subsequence $(\|x_{n_j} - P_F u\|)_{j \in \mathbb{N}}$ of $(\|x_n - P_F u\|)_{n \in \mathbb{N}}$ such that

$$\|x_{n_j} - P_F u\| < \|x_{n_j+1} - P_F u\| \text{ for all } j \in \mathbb{N}.$$

By Lemma 3.32, there exists a strictly increasing sequence $(m_k)_{k \in \mathbb{N}}$ of positive integers such that $\lim_{k \rightarrow \infty} m_k = +\infty$ and the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$\|x_{m_k} - P_F u\| \leq \|x_{m_k+1} - P_F u\| \quad \text{and} \quad \|x_k - P_F u\| \leq \|x_{m_k+1} - P_F u\|. \quad (3.32)$$

Consequently,

$$\begin{aligned} 0 &\leq \lim_{k \rightarrow \infty} (\|x_{m_k+1} - P_F u\| - \|x_{m_k} - P_F u\|) \\ &\leq \limsup_{n \rightarrow \infty} (\|x_{n+1} - P_F u\| - \|x_n - P_F u\|) \\ (\text{by (3.2)}) &\leq \limsup_{n \rightarrow \infty} (\alpha_n \|u - P_F u\| + (1 - \alpha_n) \|x_n - P_F u\| - \|x_n - P_F u\|) \\ (\alpha_n \rightarrow 0) &= \limsup_{n \rightarrow \infty} \alpha_n (\|u - P_F u\| - \|x_n - P_F u\|) = 0. \end{aligned}$$

Hence,

$$\lim_{k \rightarrow \infty} (\|x_{m_k+1} - P_F u\| - \|x_{m_k} - P_F u\|) = 0. \quad (3.33)$$

As in the **Case 1.**, we can prove that

$$\lim_{k \rightarrow \infty} \|x_{m_k} - Sx_{m_k}\| = \lim_{k \rightarrow \infty} \|x_{m_k} - Tx_{m_k}\| = 0$$

and by the demiclosedness of $I - T$ at 0 and of $I - S$ at 0, we obtain that

$$\limsup_{k \rightarrow \infty} \langle u - P_F u, x_{m_k} - P_F u \rangle \leq 0. \quad (3.34)$$

We replace in (3.31) n with m_k , then

$$\|x_{m_k+1} - P_F u\|^2 \leq (1 - \alpha_{m_k}) \|x_{m_k} - P_F u\|^2 + 2\alpha_{m_k} \langle u - P_F u, x_{m_k+1} - P_F u \rangle.$$

In particular, we get

$$\begin{aligned} \alpha_{m_k} \|x_{m_k} - P_F u\|^2 &\leq \|x_{m_k} - P_F u\|^2 - \|x_{m_k+1} - P_F u\|^2 \\ &\quad + 2\alpha_{m_k} \langle u - P_F u, x_{m_k+1} - P_F u \rangle \\ (\text{by (3.32)}) &\leq 2\alpha_{m_k} \langle u - P_F u, x_{m_k+1} - P_F u \rangle. \end{aligned} \quad (3.35)$$

Then, from (3.34), we obtain

$$\limsup_{k \rightarrow \infty} \|x_{m_k} - P_F u\|^2 \leq 2 \limsup_{k \rightarrow \infty} \langle u - P_F u, x_{m_k+1} - P_F u \rangle \leq 0.$$

Thus, from (3.32) and (3.33), we conclude that

$$\limsup_{k \rightarrow \infty} \|x_k - P_F u\|^2 \leq \limsup_{k \rightarrow \infty} \|x_{m_k+1} - P_F u\|^2 = 0,$$

i.e., $(x_n)_{n \in \mathbb{N}}$ converges strongly to $P_F u$.

□

Remark 3.2. The inequality (3.26) plays a crucial role in the proof of (iii) as the similar inequality (3.3) in Theorem 3.1 of [13]. In fact in both proofs by these inequalities some important properties of the sequence follow.

We remark that our tools are different from theirs because the techniques used in [13] seem questionable.

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F. CIANCARUSO, G. MARINO, A. RUGIANO, B. SCARDAMAGLIA: DIPARTIMENTO DI MATEMATICA ED INFORMATICA, UNIVERSITÀ DELLA CALABRIA, 87036 ARCAVACATA DI RENDE (CS), ITALY

E-mail address: cianciaruso@unical.it, gmarino@unical.it

E-mail address: rugiano@mat.unical.it, scardamaglia@mat.unical.it